value found here. It is noted that the solution of the problem is very sensitive to various parameters used such as the earth radius and the air density model.

References

[1] Bryson, A. E. and Denham, W. F., "A Steepest-Ascent Method for Solving Optimum Programming Problems," *Journal of Applied Mechanics*, Vol. 29, No. 2, June 1962, pp. 247–257.

[2] Minzner, R. A., Champion, K. S. W., and Pond, H. L., "The ARDC Model Atmosphere, 1959," TR-59-267, 1959, Air Force Cambridge Research Center, Mass., pp. 124–132.

[3] Athans, M. and Falb, P. L., *Optimal Control*, 1st ed., McGraw-Hill, New York, 1966, pp. 258–289.

[4] McCue, W. W. and Good, R. C., "A Comparison between Steepest-Ascent and Differential Correction Optimization Methods in a Problem of Bolza," Masters Degree thesis, June 1963, Naval Postgraduate School, Monterey, Calif.

[5] Hildebrand, F. B., *Advanced Calculus for Applications*, 2nd ed., Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 102–106.

Numerical Solution of Nonlinear Equations for Spinning Shallow Spherical Shells

Roy J. Beckemeyer,* T. S. David,* and Walter Eversman†

Wichita State University, Wichita, Kansas

Nomenclature

E = Young's modulus
N_{rr}^* = dimensionless radial direct stress resultant, $N_{rr}^* = \gamma Ef/\rho\omega^2R^2x$
R = shell radius of curvature
b = shell peripheral radius
f = nondimensional stress function
g = nondimensional displacement variable
r = radial coordinate
t = shell thickness
x = nondimensional radial coordinate, $x = r/b$
γ = dimensionless shell inertia loading parameter, $(3 + \nu)\rho^2\omega R^2/Et$
Δ = finite difference grid interval
λ^4 = dimensionless shell geometry parameter, $\lambda^4 = 12(1 - \nu^2)(b/R)^2/(t/b)^2$
ν = Poisson's ratio
ρ = mass per unit surface area
σ_{rr}^* = dimensionless radial bending stress resultant, $\sigma_{rr}^* = \{(3 + \nu)t/[12(1 - \nu^2)]^{1/2}\lambda^2\}[dg/dx + g/x]$

Introduction

PREPARATORY to conducting an investigation of the free vibrations of centrifugally stabilized, shallow spherical shells,[1,2] it was necessary to make a comprehensive study of the shell equilibrium stress and displacement distributions due to spin. Limitations of previous work on this problem have been surveyed in Refs. 1, 2, and 3. Some of the prior investigations[4,5] have used membrane theory which yields inconsistent results in the limiting case of the shallow spherical shell with infinite radius of curvature, the flat circular disk.

The shell configuration considered here is shown in Fig. 1. The thin, shallow spherical shell, spinning about its polar axis, has a peripheral radius b, and is fully clamped by a central hub of radius a. The outer edge of the shell is free. The defining equilibrium equations, based on Reissner's nonlinear shallow shell theory[6] may be written in terms of first central differences as a two-dimensional vector difference equation[2]

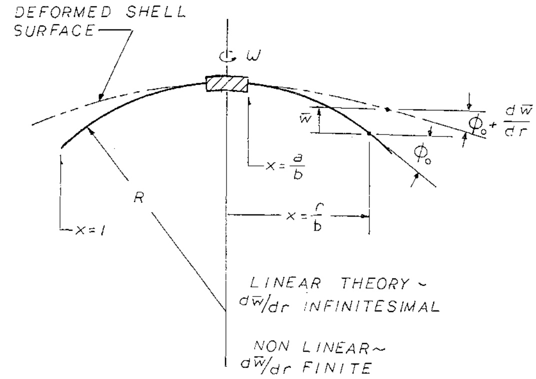

Fig. 1 Geometry and coordinate system for the spinning shallow spherical shell.

$$A_iT_{i-1} + B_iT_i + C_iT_{i+1} = D_i + E_i \tag{1}$$

$$(i = 2,3, \ldots, n - 1)$$

where

$$T_j = \begin{bmatrix} g_j \\ f_j \end{bmatrix}$$

is the unknown column vector containing g_j and f_j, the dimensionless displacement variable and stress function, respectively, and where

$$\begin{gathered} A_i = \begin{bmatrix} (1/\Delta^2 - 1/2x\Delta)x\Delta^2 & 0 \\ 0 & (1/\Delta^2 - 1/2x\Delta)x\Delta^2 \end{bmatrix} \\ B_i = \begin{bmatrix} (-2/\Delta^2 - 1/x^2)x\Delta^2 & \lambda^4x\Delta^2 \\ -x\Delta^2 & (-2/\Delta^2 - 1/x^2)x\Delta^2 \end{bmatrix} \\ C_i = \begin{bmatrix} (1/\Delta^2 - 1/2x\Delta)x\Delta^2 & 0 \\ 0 & (1/\Delta^2 - 1/2x\Delta)x\Delta^2 \end{bmatrix} \\ D_i = \begin{bmatrix} 0 \\ -x^2\Delta^2 \end{bmatrix}; \quad E_i = \begin{bmatrix} \gamma\lambda^4f_ig_i\Delta^2 \\ -\frac{1}{2}\gamma g_ig_i\Delta^2 \end{bmatrix} \end{gathered} \tag{2}$$

The boundary conditions at the hub and at the free edge of the shell may be written in terms of first forward and first

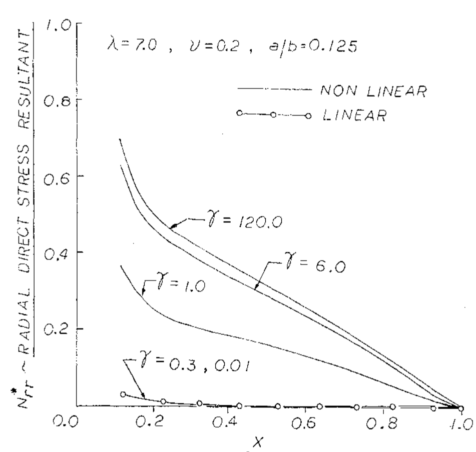

Fig. 2 Nondimensional radial direct stress resultant for a spinning shallow spherical shell, $N_{rr}^* = N_{rr}/\rho\omega b^2$. Shell geometry parameter = λ^4 = 2401.0; Poisson's ratio = ν = 0.20.

Received August 25, 1969. The research reported here was supported in part by the Applied Mathematics Branch of NASA Headquarters under Grant NGR 17-003-004.

* Graduate Student, Department of Aeronautical Engineering. Member AIAA.

† Associate Professor, Department of Aeronautical Engineering. Member AIAA.

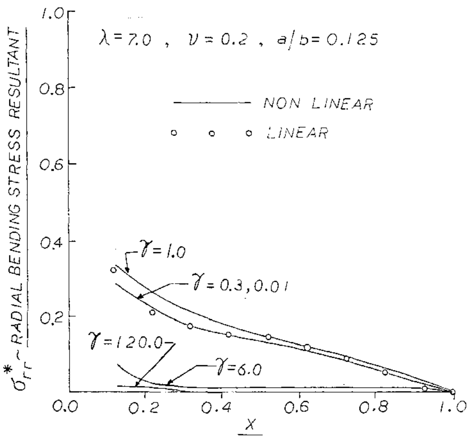

Fig. 3 Nondimensional radial bending stress resultant for a spinning shallow spherical shell, $\sigma_{rr}^* = \sigma_{rr}/\rho_0\omega b^2$. Shell geometry parameter $= \lambda^4 = 2401.0$, Poisson's ratio $= \nu = 0.20$.

backward differences, respectively, as

$$B_1T_1 + C_1T_2 = D_1; \quad A_nT_{n-1} + B_nT_n = D_n \tag{3}$$

where

$$\begin{gathered} B_1 = \begin{bmatrix} x\Delta^2 & 0 \\ 0 & (-1/\Delta - \nu/x)x\Delta^2 \end{bmatrix} \\ B_n = \begin{bmatrix} (1/\Delta - \nu/x)\Delta^2 & 0 \\ 0 & \Delta^2 \end{bmatrix} \\ C_1 = \begin{bmatrix} 0 & 0 \\ 0 & x\Delta \end{bmatrix}; \quad C_n = \begin{bmatrix} -\Delta & 0 \\ 0 & 0 \end{bmatrix} \\ D_1 = \begin{bmatrix} 0 \\ -\Delta^2x^3/(3+\nu) \end{bmatrix}; \quad D_n = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{gathered} \tag{4}$$

The finite difference expressions are based on a choice of n evenly spaced solution points with an interval Δ. Ignoring, for the moment, the nonlinear terms, E_i, in Eqs. (1), we define a sequence of linear, two-dimensional vector difference equations

$$\begin{gathered} B_1T_1 + C_1T_2 = D_1 \\ A_2T_1 + B_2T_2 + C_2T_3 = D_2 \\ A_iT_{i-1} + B_iT_i + C_iT_{i+1} = D_i \\ A_{n-1}T_{n-2} + B_{n-1}T_{n-1} + C_{n-1}T_n = D_{n-1} \\ A_nT_{n-1} + B_nT_n = D_n \end{gathered} \tag{5}$$

Method of Solution

The first approach to the solution of Eqs. (1) was the application of Archer's finite difference method for the solution of the nonlinear equations for shells of revolution.[7] Archer's technique involves solving the linear portion of the problem, Eqs. (5), and then utilizing the linear solution to evaluate the nonlinear terms. These are then treated as additional nonhomogeneous contributions to the linear equations for the next solution step. This procedure is continued until successive solutions coincide to within a specified error. Archer's chief contribution is the finite difference scheme that he has found to be appropriate for solutions to the successive linear two-point boundary value problem. In attempting to solve Eqs. (1), it was found that Archer's method failed to converge for certain combinations of the shell geometry and inertia loading parameters. A slightly modified method was then employed which uses Archer's finite difference technique but defines the successive linear two-point boundary value problem involved in the iteration in a different way. This method converged for all cases run.

The first step in solving Eqs. (1) is to obtain a solution to the linear problem defined by Eqs. (5) using Archer's approach. The first equation of (5) may be written

$$T_1 + R_1T_2 = S_1 \text{ where } R_1 = B_1^{-1}C_1 \text{ and } S_1 = B_1^{-1}D_1 \tag{6}$$

Eliminating T_1 from the second equation of (5), using the expression (6), yields

$$\begin{gathered} T_2 + R_2T_3 = S_2 \text{ where } R_2 = W_2^{-1}C_2 \\ S_2 = W_2^{-1}(D_2 - A_2S_1), \text{ and } W_2 = B_2 - A_2R_1 \end{gathered} \tag{7}$$

Continuing this procedure through the $(n-1)$th equation of (5) will yield

$$\begin{gathered} T_3 + R_3T_4 = S_3 \\ T_i + R_iT_{i+1} = S_i \\ T_{n-1} + R_{n-1}T_n = S_{n-1} \end{gathered} \tag{8}$$

where $R_j = W_j^{-1}C_j$, $S_j = W_j^{-1}(D_j - A_jS_{j-1})$, and $W_j = B_j - A_jR_{j-1}$.

The (n-1)th equation of (8) and the nth equation of (5) may be solved for T_n, yielding the following set of equations:

$$\begin{gathered} T_1 + R_1T_2 = S_1 \\ T_2 + R_2T_3 = S_2 \\ T_i + R_iT_{i-1} = S_i \\ T_{n-1} + R_{n-1}T_n = S_{n-1} \\ T_n = S_n \end{gathered} \tag{9}$$

where $S_n = W_n^{-1}(D_n - B_nS_{n-1})$, and $W_n = C_n - A_nR_{n-1}$.

Note that Eqs. (9) have essentially been reduced to an "upper triangular" form; thus the solution of the linear equations may be considered a Gauss reduction technique. The nth equation of (9) yields T_n directly as

$$T_n = (C_n - A_nR_{n-1})^{-1}(D_n - A_nS_{n-1}) \tag{10}$$

Back substitution into the $n - 1$ remaining equations yields values for all the T_j, $(j = n, n-1, n-2, \ldots, 3,2,1)$. We now have a solution, T_j (1), to the linear portion of (1). The true solution to the total nonlinear problem defined by Eqs. (1) may be written

$$T_j = T_j(1) + T_j(2) \tag{11}$$

where the $T_j(2)$ are unknown remainder terms. Substituting (11) into (1), the governing equations for the remainder terms become

$$A_i(2)T_{i-1}(2) + B_i(2)T_i(2) + C_i(2)T_{i-1}(2) = D_i(2) + E_i(2) \tag{12}$$

where $A_i(2) = A_i$, $C_i(2) = C_i$,

$$\begin{gathered} B_i(2) = B_i + \begin{bmatrix} -\Delta^2\gamma\lambda^4f_i & -\Delta^2\gamma\lambda^4g_i \\ \Delta^2\gamma g_i & 0 \end{bmatrix} \\ D_i(2) = \begin{bmatrix} f_ig_i\Delta^2\gamma\lambda^4 \\ -\frac{1}{2}g_ig_i\Delta^2\gamma \end{bmatrix} \\ E_i(2) = \begin{bmatrix} \gamma\lambda^4f_i(2)g_i(2)\Delta^2 \\ -\frac{1}{2}\gamma g_i(2)g_i(2)\Delta^2 \end{bmatrix} \end{gathered}$$

and where $g_i = g_i(1), f_i = f_i(1)$ are known functions of x.

Repeating the linear solution procedure, the solution to the linear portion of (12), $T_j(3)$, is used to express the total solution of (12) as

$$T_j(2) = T_j(3) + T_j(4) \tag{13}$$

where the $T_j(4)$ are again some unknown remainder terms. After m such iterations, the equations for the unknown remainder terms $T_j(2m)$ are

$$A_i(2m)T_{i-1}(2m) + B_i(2m)T_i(2m) + C_i(2m)T_{i-1}(2m) = D_i(2m) + E_i(2m) \tag{14}$$

with

$$A_i(2m) = A_i, \; C_i(2m) = C_i$$

$$B_i(2m) = B_i + \begin{bmatrix} -\Delta^2\gamma\lambda^4 f_i & -\Delta^2\gamma\lambda^4 g_i \\ \Delta^2\gamma g_i & 0 \end{bmatrix}$$

$$D_i(2m) = \begin{bmatrix} f_i g_i \Delta^2\gamma\lambda^4 \\ -\frac{1}{2} g_i g_i \Delta^2 \gamma \end{bmatrix}$$

$$E_i(2m) = \begin{bmatrix} \gamma\lambda^4 f_i(2m) g_i(2m)\Delta^2 \\ -\frac{1}{2}\gamma g_i(2m) g_i(2m)\Delta^2 \end{bmatrix}$$

where now

$$T_j = \begin{bmatrix} g_j \\ f_j \end{bmatrix} = T_j(1) + T_j(3) + \ldots + T_j(2m-1) = \begin{bmatrix} g_j(1) + g_j(3) + \ldots + g_j(2m-1) \\ f_j(1) + f_j(3) + \ldots + f_j(2m-1) \end{bmatrix} \quad (15)$$

The iteration is repeated until the remainder terms $T_j(2m)$ become small compared with the accumulated solution $T_j(1) + T_j(3) + \ldots + T_j(2m-1)$. The main advantage of the modified method is that after the first iteration some features of the nonlinearity are included in the linear equations, whereas in Archer's approach the particular nature of the nonlinear terms is relatively unimportant at a given solution step.

Reference 3 contains a comprehensive summary of the stresses and displacements in a spinning shallow spherical shell obtained by using Reissner's linear[8] and nonlinear[6] theories for a significant range of shell geometry and inertia loading parameters. Figures 2 and 3 show the variation of two of the stress components, the nondimensional radial direct stress and radial bending stress resultants, respectively, for a shell with fairly substantial curvature.

Note in Fig. 2 that for low γ or low inertia loading, the nonlinear and linear solutions nearly coincide whereas for large γ the solution is nearly that of a flat disk. For small γ, a membrane stress distribution is approached. As the inertia load increases, the shell progressively flattens out and the stress distribution alters radically, until, finally, the stress distribution is essentially the same as if the shell had been flat to start with. That is, the stress required to overcome the initial curvature is small compared to the added stress built up after the disk is almost flat.

As shown in Fig. 3, nonlinear effects appear in the bending stress resultant at even small inertia loadings. The effect of increasing γ is to decrease the bending stress. For high γ the residual bending stress becomes small as the membrane effects predominate.

References

[1] Beckemeyer, R. J. and Eversman, W., "The Free Vibrations of a Spinning Centrally-Clamped Shallow Spherical Shell," Aeronautical Rept. 69-1, Dec. 1969, Dept. of Aeronautical Engineering, Wichita State Univ.

[2] Beckemeyer, R. J., "The Free Vibrations of a Spinning Centrally-Clamped Shallow Spherical Shell," M.S. thesis, May 1969, Department of Aeronautical Engineering, Wichita State University.

[3] Eversman, W., "Some Equilibrium and Free Vibration Problems Associated with Centrifugally Stabilized Disk and Shell Structures," CR-1178, Dec. 1968, NASA.

[4] Johnson, M. W., "On the Dynamics of Shallow Elastic Membranes," *Proceedings of the Symposium on the Theory of Thin Elastic Shells*, edited by W. T. Koiter, North-Holland, Amsterdam, 1960, pp. 281–300.

[5] Simmonds, J. G., "The Dynamics of Large Rotating Membrane-Like Structures," Paper 60-87, presented at the IAS National Summer Meeting, Los Angeles, 1960, Institute of the Aeronautical Sciences.

[6] Reissner, E., "Stresses and Small Displacements of Shallow Spherical Shells, I," *Journal of Mathematical Physics*, Vol. 25, 1946, pp. 80–85.

[7] Archer, R. R., "On the Numerical Solution of the Nonlinear Equations for Shells of Revolution," *Journal of Mathematical Physics*, Vol. 41, 1962, pp. 1965–1978.

[8] Reissner, E., "Stresses and Small Displacements of Shallow Spherical Shells, II," *Journal of Mathematical Physics*, Vol. 25, 1946, pp. 279–300, and Vol. 27, 1948, pp. 240.

Unsteady Three-Dimensional Stagnation-Point Flow

Daniel F. Jankowski* and John M. Gersting Jr.†
Arizona State University, Tempe, Ariz.

Nomenclature

f, F, g, G	=	nondimensional velocity functions
n	=	$(\beta/\alpha)^{1/2}$
p	=	pressure
P	=	nondimensional pressure function
t	=	time
u, v, w	=	x, y, z velocity components, respectively
x	=	coordinate tangent to surface
y	=	coordinate normal to surface
z	=	coordinate normal to x, y plane
α, β, γ	=	constants in velocity components
η, ξ	=	similarity variables in y and t
λ	=	acceleration parameter
ν	=	kinematic viscosity
ρ	=	density

Introduction

In a recent Note, Williams[1] found exact solutions to the Navier-Stokes equations for a special class of unsteady incompressible flows in the vicinity of either a two-dimensional or an axisymmetric stagnation point. These solutions are obtained by solving a single nonlinear, ordinary differential equation. In the present Note, we show, by extending the work of Howarth[2] on the boundary layer in three-dimensional flow, that exact solutions to the Navier-Stokes equations are possible for a similar class of unsteady incompressible flows in the vicinity of a stagnation point on a general (three-dimensional) surface. In this general case, the problem reduces to two simultaneous nonlinear, ordinary differential equations containing a parameter; for particular values of this parameter, the two-dimensional and axisymmetric cases are regained.

Analysis

Consider an infinite plate in the x-z plane. There is a stagnation point at $x = y = z = 0$. The class of three-dimensional (steady) flows considered by Howarth is characterized by $u \sim x$, $w \sim z$, and $v \sim y$ as $y \to \infty$, and is determined by solving the Navier-Stokes equations exactly. The two-dimensional unsteady flow considered by Williams is characterized by $u \sim x/(1 + \lambda t)$ and $v \sim y/(1 + \lambda t)$ as $y \to \infty$. His unsteady axisymmetric case exhibits a similar time dependence. Here, guided by the aforementioned considerations, we incorporate the unsteady similarity variable of Williams' work with the three-dimensional form used by Howarth. Thus, for a three-dimensional, unsteady stagnation-point flow, the velocity components are assumed to be

$$u = \alpha x F'(\xi)/(1 + \lambda t), \; w = \beta z G'(\xi)/(1 + \lambda t) \quad (1)$$

and

$$v = -[\nu/\gamma(1 + \lambda t)]^{1/2}\{\alpha F(\xi) + \beta G(\xi)\} \quad (2)$$

Received August 15, 1969.

* Associate Professor, Department of Mechanical Engineering.

† Faculty Associate, Department of Engineering Mechanics. Student Member AIAA.